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Marking Scheme -Mathematical Olympiad – Stage I – 2015

Note: All alternate solutions are to be accepted at par. Please asses what a student know in place of what he doesn't know.

1. **When the tens digit of a three digit number $\overline{abc}$ is deleted , a two digit number $\overline{ac}$ is formed . How many numbers $\overline{abc}$ are there such that**

   $\overline{abc}=9\,\overline{ac}+4c$.

Soln. Let the three digit no. $\overline{abc}$ be 100a+10b+c and similarly $\overline{ac}$ =10a+c

2 marks

As per given equation 100a+10b+c= 9*(10a+c)+4c

which gives on simplification

3 marks

5(a+b)=6c

Since a,b,c are integers between 0 and 9 and $a\neq 0$, The only possibility is a+b=6 and c=5

3 marks

so by taking b=6-a, six possible numbers arise by taking a=1,2,3,4,5,6

So there are six three digit numbers.

2 marks

2. **Let $p(x) = x^2+bx+c$ where b and c are integers. If p(x) is a factor of both x^4+6x^2+25 and $3x^4+4x^2+28x+5$, find the value of p(1)?**

   Soln.

   $X^4+6x^2+25=(x^4+10x^2+25)-4x^2$ — 1 mark

   ⇨ $(x^2+5)^2-(2x)^2$ — 2 marks

   ⇨ $(X^2+5+2x)(x^2+5-2x)$

   ⇨ So p(x) is either x^2+5+2x or x^2+5-2x — 2 marks

   ⇨ By long division we find that only x^2-2x+5 is factor of $3x^4+4x^2+28x+5$.

   ⇨ So $3x^4+4x^2+28x+5= (x^2-2x+5)(3x^2+6x+1)$ — 3 marks

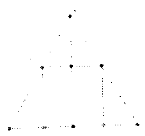
   ⇨ $p(x)= x^2-2x+5$

   ⇨ p(1)=4 — 2 marks

3. A square is inscribed in an equilateral triangle. Find the ratio of area of the square to that of the triangle.

Solution:

Fig 1 mark



Let the side of square be 'a' so AG= $\frac{\sqrt{3}}{2}a$ and GD=a

3 marks

So AD=AG+GD=$(\frac{\sqrt{3}}{2}+1)$ a -------------------- (1)

If 'b' be the side of equilateral triangle . Altitude of triangle is AD= $\frac{\sqrt{3}}{2}$b---------- (2)

From 1 and 2, $(\frac{\sqrt{3}}{2}+1)\,a = \frac{\sqrt{3}}{2}\,b$

So a= $\frac{\sqrt{3}}{2+\sqrt{3}}b$

3 marks

Now area of square = $a^2 = \frac{3}{7+4\sqrt{3}}b^2$ (3)

2 marks

Area of Triangle= $\frac{\sqrt{3}}{4}b^2$ (4)

Ratio of areas of Square to Triangle : $\frac{3}{7+4\sqrt{3}}b^2 : \frac{\sqrt{3}}{4}b^2$

2 marks

$\frac{\sqrt{3}}{7+4\sqrt{3}}:\frac{1}{4} = 4\sqrt{3} : 7+4\sqrt{3}$

4. (a) **Prove that** $\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\ldots\ldots+\frac{1}{98}-\frac{1}{99}+\frac{1}{100}>\frac{1}{5}$

$(\frac{1}{2}-\frac{1}{3})+(\frac{1}{4}-\frac{1}{5})+.........+(\frac{1}{98}-\frac{1}{99})+\frac{1}{100}$

3 marks

Soln. $\frac{1}{6}+\frac{1}{20}+..............+\frac{1}{9702}+\frac{1}{100}$

⇨ Now Let us find $\frac{1}{6}+\frac{1}{20}=\frac{13}{60}=\frac{12+1}{60}=\frac{12}{60}+\frac{1}{60}=\frac{1}{5}+\frac{1}{60}>\frac{1}{5}$

1 mark

So $(\frac{1}{2}-\frac{1}{3})+(\frac{1}{4}-\frac{1}{5})+.........+(-\frac{1}{99}+\frac{1}{100})>\frac{1}{5}$

1 mark

4 (b) **Find the largest prime factor of** $3^{12}+2^{12}-2.6^{6}$

Soln

$3^{6})^{2}+(2^{6})^{2}-2.3^{6}.2^{6}$

$=> (3^{6}-2^{6})^{2}=\{(3^{3})^{2}-(2^{3})^{2}\}^{2}$

3 marks

$=> \{(3^{3}+2^{3})(3^{3}-2^{3})\}^{2}$

$=> \{(3-2)(9+4+6)(3^{3}+2^{3})\}^{2}$

1 mark

$=> \{(19)(3+2)(9+4-6)\}^{2}$

$=> \{(19)(5)(7)\}^{2}$

1 mark

So largest prime factor is 19

5. Surface area of a sphere A is 300 % more than the surface area of another sphere B. If the volume of sphere B is p% less than the volume of sphere A,find the value of 'p'.

Soln. Let the radius of sphere B be 'r'

So its surface area = $4\pi r^{2}$

Let surface area of A be S which is 300% more than that of B

$$\frac{(s-4\pi r^2)}{4\pi r^2}X100 = 300$$ 2 marks

$$=> s-4\pi r^2 = 12\pi r^2$$

So we get

$$=> s = 16\pi r^2$$ 1 mark

Let radius of sphere A be R so $4\pi R^2 = 16\pi r^2$

⇨ We get R=2r 2 marks

⇨ Now volume of sphere A = $\frac{4}{3}\pi R^3$

⇨ $\frac{4}{3}\pi(2r)^3 = \frac{32}{3}\pi r^3$ 2 marks

Volume of sphere B be = $\frac{4}{3}\pi r^3$ which is p% less than that of A

So $$\frac{\frac{32\pi r^3}{3} - \frac{4}{3}\pi r^3}{\frac{32}{3}\pi r^3} x100 = p$$

$$\frac{\frac{28}{3}\pi r^3}{\frac{32}{3}\pi r^3} x100 = p$$ 2 marks

⇨ $$=> p = \frac{700}{8} = 87.5$$ 1 mark

6. ABC is an isosceles triangle in which AB=AC=25 cm and BC=14 cm Find the difference of the circum-radius and in- radius of the triangle.

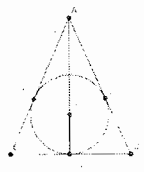
Soln. Draw AD $\perp$ BC as AB=AC so D will be mid point of BC

Now we have $AB^2 = AD^2 + BD^2$

$$=> (25)^2 = (AD)^2 + (7)^2$$

2marks

$$=> AD = 24$$

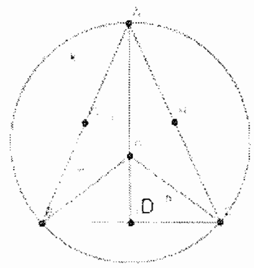


Let O be the In-centre of the triangle. So OD is in-radius.

Now area of $ABC = \frac{1}{2} BCXAD = \frac{1}{2} X14X24 = 168cm^2$ -- (1) 2 marks

Also are of Triangle ABC = $\frac{1}{2}(AB + BC + AC) * r$ **where r is in-radius.**

= 32*r cm^2 ----------------------



From (1) and (2) we get r= $\frac{21}{4} cm.$ 2 marks

. Let P be the Circumcentre of the Triangle ABC.

Let PA=PB=PC=R = the Circum-radius

So PD =24-R

Now from Triangle PBD , $PB^2=PD^2+BD^2$

- ⇨ $R^2=(24-R)^2+7^2$
- ⇨ $R^2=576-48R+R^2+49$
- ⇨ $R=\frac{625}{48}$

3 marks

Now required difference of two radii= $(\frac{625}{48}-\frac{21}{4})$

$=\frac{373}{48}cm.$

1 mark

7. AB and BC are two equal chords of a circle of length $2\sqrt{5}$ cm each . If radius of the circle is 5 cm, find the length of the chord AC .
   Soln. Let O be the centre of the circle Join OA, OB and OC .
   As AB=BC so OB will be the perpendicular bisector of AC

fig- 1 mark

Let OB intersect AC at D and perpendicular to AD, Ler OD=x, so BD=5-x

Now from Triangle OAD, $OA^2=AD^2+OD^2$ 3 marks

- ⇨ $25=AD^2+x^2$ (1)
- ⇨ From triangle BDA we get , $AB^2=AD^2+BD^2$
- ⇨ $(2\sqrt{5})^2 = AD^2 + (5-x)^2$ (2) 3 marks
- ⇨ From (1) and (2) we get x=3
- ⇨ Putting this value we get AD=4
- ⇨ So AC=2AD=8 cm.

3 marks

8. Two dice are thrown simultaneously. Find the sum of the probability of "getting a prime number as a sum" and probability of "getting a doublet of prime numbers."

Soln. Total No. of outcomes=6X6=36

Outcomes for getting a prime number as sum are:

(1,1),(1,2),(1,4),(1,6),(2,1),(2,3),(2,5),(3,2),(3,4),(4,1),(4,3),(5,2),(5,6),(6,1) and (6,5) = 15 outcomes 3 marks

So probability (Sum as Prime number) $=\frac{15}{36}$ 1 Mark

Further , Prime doublets are (2,2),(3,3),(5,5) 2 marks

So probability of a doublet as prime No. = $\frac{3}{36}$ 2 marks

Now required probability = $\frac{15}{36}+\frac{3}{36}=\frac{1}{2}$ 2 marks

9. A person starts from a place P towards another place Q at a speed of 30 km/ h. After every 12 minutes, he increases his speed by 5 km/h. If the distance between P and Q is 51 km., find the time taken by him to cover the whole distance.

Soln. Let he covers the distance in n intervals of 12 minutes each.

In the first interval speed is 30 km/h so distance travelled= 30X12Min=6km.

In the 2nd interval speed is 35km/h, so distance= 35X12min= 7km.

And so on............

2 marks

So total distance travelled in n intervals= 6+7+8+..............up to n terms.

⇨ $\frac{n}{2}\{2\times 6+(n-1)\times 1\}$ 2 marks

⇨ $\frac{n^2+11n}{2}km$ =51 km (given) 2 marks

⇨ $n^2+11n-102=0$

⇨ on simplification it gives n= - 17(rejected) and 6 3 marks

⇨ so there are 6 intervals of 12 min each.

⇨ Hence total time 6X12=72 minutes. 1 mark

10. Solve for 'x' ;

$$4(x-\frac{1}{x})^2+8(x+\frac{1}{x})=29$$

Soln. The equation can be written as

$$4(x^2+\frac{1}{x^2}-2)+8(x+\frac{1}{x})-29=0$$

$$=> 4(x^2+\frac{1}{x^2})+8(x+\frac{1}{x})-37=0$$

3 marks

Let $x+\frac{1}{x}=y$

So the equation reduces to $4(y^2-2)+8y-37=0$

⇨ $4y^2+8y-45=0$

⇨ $(2y-5)(2y+9)=0$

⇨ Y=5/2 or y=-9/2

3 marks

⇨ When y=5/2 we get $x+\frac{1}{x}=\frac{5}{2}$

⇨ $2x^2-5x+2=0$

⇨ (x-2)(2x-1)=0 this gives x=2 or ½

2 marks

⇨ Taking y=-9/2

⇨ We get x= $\frac{-9\pm\sqrt{65}}{4}$

2 marks